002-Two-dimensional Geometry

Given a heptagon, with side length S and area A. If we double S, what is the area of the larger heptagon? A: Multiply the area by 4

Discrete Objects:

Something with clearly defined bounderies

Sets:

Set operations include intersection (∩), union (∪), and complement (???) - form a new *set*

Order does not matter EXCEPT when ellipses are used. Ex: *U = { a, b, c, . . ., x, y, z }*

Pre-difined Sets:

N = natural numbers = { 1, 2, 3, 4, 5, . . . }

Z = Integers = { 0, +- 1, +- 2, +-3, . . . }

N_0 = Natural Numbers and 0 = { 0, 1, 2, 3, 4, 5, . . . }

Q = Rational Numbers = { $r \mid r = \frac{a}{b};\ a \in Z,\ b \in N$

Ex: E = { 2, 4, 6, 8, . . . } → $E = \{ n | n = 2k;\ k \in N \}$

Subsets:

$A \subseteq B$ = A is a subset of B: Everything in A is in B

$A \subset B$ = A is a proper subset of B: Everything in A is in B but something in B is not in A

Discrete Objects:

A set within a set

The power set of A is a set consisting of all subsets ofA

A = { a, b } P(A) = { Ø, { a }, { b }, { a, b } }

Sequences:

A list of discrete objects for which order and repetition matter

Graphs:

Help realize relationships between discrete objects

Ex: if we define a graph using two sets *V* and *E*. Set *V* = { a, b, c, d, e, f },

Set *E* contains the edges = { (A, C), (A, D), (C, D), (B, D), . . . }

Conflict Graphs show contention or competition between discrete objects

Cliques and Wars:

A 3-person Clique is a set of three vertices connected with three edges

A 3-person War is a set of three vertices with no shared edges

Proofs:

A rigorous way of convincing you or others of something precise

Proof by contradiction - find the contradiction that makes the statement impossible

Exhaustive (case by case) proof - go through every data point and check

1) Transform the problem into a more workable form

Algorithm for making and proving a claim:

1) Precisely state the right thing to prove
2) Prove the claim
3) Check the proof for correctness

Axioms, conjectures, and Theorems:

An axiom is a self-evident statement that is asserted as true without proof

A conjecture is a claim that is believed to be true, but not trusted without additional proof

A theorem is a proven conjecture (i.e. a proven truth)

Logical Operators:

NOT: ¬

AND: ∧

OR: ∨

IF…THEN: →

A direct proof of an implication:

Given a claim in the form $p \rightarrow q$, we can consider using a direct proof as follows

Proof. We prove the implication using a direct proof

1. Start by assuming that the statement claimed in *p* is true
2. Restate your assumption in mathematical terms, as necessary
3. Use mathematical and logical derivations to relate your above assumptions to *q*
4. Argue that you have shown that *q* must be true
5. End by concluding that *q* is true

A contraposition proof of an implication:

Given a claim in the form of $p \rightarrow q$, we can consider using contraposition as follows:

Proof. We prove the implication using contraposition

1. Start by assuming the statement claimed in *q* is false
2. Restate your assumption in mathematical terms, as necessary
3. Use mathematical and logical derivations to relate your above assumption to *p*
4. Argue that you have shown that *p* must be false

Equivalence (IFF) is stronger than implication:

Claims sometimes involve equivalence between propositions *p* and *q*

p if and only if *q*.

In such compound statements, either *p* and *q* are both true or they are both false…

Proof by Induction:

Given the claim P(n), we construct a proof by induction ti show P(n) holds for all n >= n_0

Proof. We use induction to prove $\forall n \geq n_0 : P(n)$

1. Show that $P(N_0)$ is T
2. Show that P(n) → P(N+1) for a general $n \geq n_0$

3. Conclude at the end that P(n) holds for all $n \geq n_0$

Strong Induction:

Consider P(n), the Fundamental Theorem of Arithmetic, which states that for all n >= 2, we can write it as the product of two or more prime numbers

1. Smaller values do help: 12180 = 60 * 203 , or P(60) ∧ P(203) → P(12180)
2. From this, P(4) ∧ P(15) → P(60) and P(7) ∧ P(29) → P(203) and so on

Make a stronger case that all values up to n are all prime numbers

Ex:

Proof. We prove by induction that Q(n) is T for n >= 2.

1. Base case: Q(2) = P(2), ie 2 is a product of prime numbers 2 * 1
2. Induction step: We show that Q(n) → Q(n+1) for all n >= 2 via a direct proof

Assume Q(n) is T: each of 2, 3…., n is a product of prime numbers

We must prove that Q(n+1) is a product of primes

By our induction hypothesis, Q(n), observe that 2, 3, …., n are products of primes

Case 1: n + 1 is prime. In this case, we have nothing more to prove

Case 2: n + 1 is not prime, so n+1 = kl, where 2 <= k, l <= n. From our induction hypothesis, both P(k) and P(l) are T, which shows k ad l to be products of primes.

Therefore, n + 1 = kl is a product of primes and Q(n + 1) is shown to be T

L-Tile land problem (Structural Induction):

Given an unlimited supply of L shaped tiles, can we tole a 2^n x 2^n square patio, ignoring one center tile?

A LOT IS SKIPPED HERE

$A|B$ means that either the thing on the left is a multiple of the thing on the right

AKA B mod(A) = 0

Modular Arithmetic:

do it out and check for pattern

We define integers a and b to be congruent modulo integer d as follows:

A = b mod(d) → d|(a-b), ie. (a-b)=k * d for some integer k

Eg. 41 = 79 mod(19) since 41-79 = k*19 with k = -2

Reflexive - a = a mod(a)

Symmetric - a = b mod(d) → a mod(d)

Transative - if a = b and b = c mod(d), then a = c mod(d)

Is 136 equiv 592 mod(12)?

rem(136, 12) = 4 and rem(592, 12) = 12

12 | (136 - 592)